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On the Purity of the Limiting Gibbs State for the Ising Model on the Bethe Lattice

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We give a proof that for the Ising model on the Bethe lattice, the limiting Gibbs state with zero effective field (disordered state) persists to be pure for temperature below the ferromagnetic critical temperature T_c^F until the critical temperature T_c^{SG} of the corresponding spin-glass model. This new proof revises the one proposed earlier.

KEY WORDS: Pure Gibbs state; Ising–Bethe model; spin-glass.

1. The question of the possibility of the disordered limiting Gibbs state (i.e., with zero effective field) of the ferromagnetic Ising model on the Bethe lattice to be pure below the ferromagnetic critical temperature T_c has been formulated as an open problem in refs. 2 and 3 (some nonhomogeneous pure Gibbs states were constructed in refs. 4 and 5). In ref. 1 it was proved that the disordered phase is pure up to the spin-glass critical temperature T_c^{SG}, but as was found out later in correspondence between the author of ref. 1 and Hans-Otto Georgii, in the recursive formula used in ref. 1 for the probability measures ν_x [see the formula (3.7) below], the multiplier $a(h_v)$ was missed both in numerator and denominator. A corrected version of the proof[6] was based on an inequality valid for the Bethe lattice τ^k of degree $k \leqslant 6$. In the present note we give a new proof valid for all k.

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2. Let τ^k be the Bethe lattice of degree $k \geqslant 2$ such that exactly $(k+1)$ edges come out of any vertex. Then the Ising model is defined by the Hamiltonian

$$H(\sigma) = -\sum_{x,y} J_{xy}\sigma(x)\,\sigma(y) \tag{2.1}$$

where the sum is over pairs of nearest neighbors x, y and $\sigma(x) = \pm 1$. The ferromagnetic model corresponds to $J_{xy} = J > 0$, while the spin-glass model corresponds to $J_{xy} = \pm J$ $(J > 0)$, where $\{J_{xy}\}$ are independent random variables with $\Pr\{J_{xy} = +J\} = \Pr\{J_{xy} = -J\} = 1/2$ for any pair xy. Denote

$$\theta = \tanh \beta, \qquad \beta = J/k_{\mathrm{B}} T \tag{2.2}$$

where T is the temperature. Then for the ferromagnetic model the critical value is $\theta_c^{\mathrm{F}} = 1/k$ (see, e.g., ref. 3), while for the spin-glass model $\theta_c^{\mathrm{SG}} = 1/\sqrt{k}$; see, e.g., ref. 7. The main result of this note is the following theorem.

Theorem. The limiting Gibbs state with zero effective field (disordered phase) is pure for $0 < \theta \leqslant \theta_c^{\mathrm{SG}}$.

Remark. For $0 < \theta \leqslant \theta_c^{\mathrm{F}}$ the limiting Gibbs state is unique, and so trivially pure. On the other hand, for $\theta > 1/\sqrt{k}$ the disordered phase is not pure.[2,4] Thus our main result concerns the interval $[1/k, 1/\sqrt{k}]$.

3. Let $\tau^k = (V, L, i)$ be the Cayley tree of order k with a root vertex $x_* \in V$. Here V is the set of vertices, L is the set of edges, and i is the incidence function which corresponds to each edge $l \in L$ its endpoints $x_1, x_2 \in V$. There is a distance $d(x, y)$ on V which is the length of the minimal path from x to y, assuming that the length of any edge is 1. Denote by

$$W_n = \{x \in V\colon d(x_*, x) = n\}$$

the sphere of radius n on V, and by

$$V_n = \{x \in V\colon d(x_*, x) \leqslant n\}$$

the ball of radius n, so that

$$V_n = \bigcup_{m=0}^{n} W_m$$

For any $x \in W_n$, $n = 0, 1, 2,\ldots$, denote by

$$S(x) = \{y \in W_{n+1}\colon d(x, y) = 1\}$$

Let $\{h(x), x \in V\}$ be a set of real numbers satisfying for each $x \in V$ the recursive relation

$$h(x) = \sum_{y \in S(x)} f_\theta(h(y)), \qquad f_\theta(h) = \operatorname{artanh}[\theta \tanh(h)] \tag{3.1}$$

Define the Gibbs probability distribution on the configurational space

$$\Sigma(V_n) = \{\sigma_n = \{\sigma(x) = \pm 1, x \in V_n\}\}$$

by the formula

$$\mu_n(\sigma_n) = Z_n^{-1} \exp\left[\beta \sum_{x, y \in V_n : d(x, y) = 1} \sigma(x)\ \sigma(y) + \sum_{x \in W_n} h(x)\, \sigma(x)\right] \tag{3.2}$$

Then (3.1) implies the consistency of μ_n for different n, so that there exists a Gibbs measure μ on the infinite configurational space

$$\Sigma(V) = \{\sigma = \{\sigma(x) = \pm 1, x \in V\}\}$$

whose finite-dimensional distributions are μ_n. The function $h(x)$, $x \in V$, which satisfies (3.1) is called an effective field. Remark that $h(x) \equiv 0$ satisfies (3.1), so that the finite-dimensional distributions

$$\mu_n^{\#}(\sigma_n) = Z_n^{-1} \exp\left[\beta \sum_{x, y \in V_n : d(x, y) = 1} \sigma(x)\, \sigma(y)\right] \tag{3.3}$$

are consistent and generate a Gibbs measure $\mu^{\#}$, the disordered phase.

To prove that $\mu^{\#}$ is pure, we will prove that the spins $\{\sigma(x), x \in V_n\}$ and $\{\sigma(x), x \in W_N\}$ are asymptotically independent with respect to $\mu^{\#}$, when $N \to \infty$ and n is fixed. To that end we first fix $N > 0$ and define recursively for every $x \in V_{N-1}$ a random variable h_x such that

$$h_x = \sum_{y \in S(x)} f_\theta(h_y), \qquad \forall x \in V_{N-2} \tag{3.4}$$

with the initial data

$$h_x = \beta \sum_{y \in S(x)} \sigma(y), \qquad \forall x \in W_{N-1} \tag{3.5}$$

Equations (3.4), (3.5) define h_x as a function of $\sigma(y)$, $y \in W_N \cap V_x$, where V_x is the subtree growing from x. It is to be noted that in general the random variables $\{h_x, x \in V_{N-1}\}$ depend on N.

Lemma 3.1. For every $n \leqslant N-1$, the joint distribution of $\sigma_n = \{\sigma(x),\ x \in V_n\}$ and $h^{(n)} = \{h_x, x \in W_n\}$ with respect to $\mu^{\#}$ is given by

$$\mu^{\#}(\sigma_n, h^{(n)}) = Z^{-1} \exp\left[\beta \sum_{x,\, y \in V_n : d(x,y)=1} \sigma(x)\, \sigma(y) + \sum_{x \in W_n} h_x \sigma(x)\right] \times \prod_{x \in W_n} \nu_x(h_x) \tag{3.6}$$

where the probability distributions $\nu_x(h)$ are defined by the recursive equations

$$\nu_x(h) = \frac{\sum_{h_y,\dots,h_z : f(h_y) + \cdots + f(h_z) = h} \left[\prod_{v \in S(x)} a(h_v)\, \nu_v(h_v)\right]}{\sum_{h_y,\dots,h_z} \left[\prod_{v \in S(x)} a(h_v)\, \nu_v(h_v)\right]}, \quad \forall x \in V_{N-2} \tag{3.7}$$

where

$$S(x) = (y,\dots, z), \qquad f(h) = f_\theta(h), \qquad a(h) = [1 + (1-\theta^2) \sinh^2 h]^{1/2} \tag{3.8}$$

and by the initial condition

$$\nu_x(h) = Z^{-1} \sum_{\sigma(y),\dots,\sigma(z) : \beta[\sigma(y) + \cdots + \sigma(z)] = h} 1, \qquad \forall x \in W_{N-1} \tag{3.9}$$

Remark. Lemma 3.1 allows the following extension. Let μ be a Gibbs measure generated, according to (3.2), by some set of real numbers $\{h(x),\ x \in V\}$ which satisfy the consistency equations (3.1), and let $\{h_x, x \in W_{N-1}\}$ be random variables defined through (3.4) and (3.5). Then the joint distribution $\mu(\sigma_n, h^{(n)})$ has the form (3.6) with probability distributions $\nu_x(h)$ defined by the recursive equations (3.6) and by the initial condition

$$\nu_x(h) = Z^{-1} \sum_{\sigma(y),\dots,\sigma(z) : \beta[\sigma(y) + \cdots + \sigma(z)] = h} \exp[h(y)\, \sigma(y) + \cdots + h(z)\, \sigma(z)], \qquad \forall x \in W_{N-1} \tag{3.10}$$

When $h(x) \equiv 0$, (3.10) reduces obviously to (3.9). If, in addition, the coupling constant J_{xy} depend on xy, then again the formulas (3.2)–(3.8), (3.10) are valid with

$$\beta = J_{xy}/k_{\mathrm{B}} T, \qquad \theta = \tanh(J_{xy}/k_{\mathrm{B}} T)$$

and some natural modifications in these formulas. The proof of this extension of Lemma 3.1 is similar to the proof of Lemma 3.1.

Proof of Lemma 3.1. The proof of (3.7) is based on the following identity:

$$\sum_{\sigma(y)=\pm 1} \exp[\beta\sigma(x)\sigma(y)+h_y\sigma(y)] = Za(h_y)\exp[f_\theta(h_y)\sigma(x)] \quad \text{if } \sigma(x)=\pm 1 \tag{3.11}$$

where $Z=\cosh\beta$. To prove this identity, notice that according to $\sigma(x)=\pm 1$, it reduces to two identities:

$$\exp(\beta+h_y)+\exp(-\beta-h_y)=Za(h_y)\exp f_\theta(h_y)$$

$$\exp(-\beta+h_y)+\exp(\beta-h_y)=Za(h_y)\exp[-f_\theta(h_y)]$$

If we divide the first identity by the second one, we obtain

$$\frac{\exp(\beta+h_y)+\exp(-\beta-h_y)}{\exp(-\beta+h_y)+\exp(\beta-h_y)}=\exp[2f_\theta(h_y)]$$

which follows easily from the formula $\exp(2\,\text{artanh}\, z)=(1+z)/(1-z)$. If we multiply the first identity by the second one, we obtain

$$\cosh^2\beta+\sinh^2 h_y = Z^2a^2(h_y)$$

which follows easily from the formula $a^2(h)=1+\sinh^2 h/\cosh^2\beta$. Thus (3.11) is proved. Multiplying (3.11) over $y\in S(x)$, we obtain (3.7) by induction. Equation (3.9) follows directly from (3.3). Lemma 3.1 is proved. ∎

4. By (3.9) and (3.7) the measures $\nu_x(h)$ coincide for all $x\in W_n$, so we will denote then by $\nu_n(h)$. By (3.9) and (3.7) $\nu_n(h)$ is symmetric, hence

$$\sum_h \sinh h\nu_n(h)=0$$

Let

$$D_n=\sum_h \sinh^2 h\nu_n(h)$$

Lemma 4.1. If $k\theta^2\leqslant 1$, then

$$D_{n-1}\leqslant k\theta^2 D_n \tag{4.1}$$

Proof. It is convenient to rewrite (3.4) and (3.7) in the form

$$h=\sum_{j=1}^{k} f(h_j), \qquad f(h)=f_\theta(h)$$

and

$$v_{n-1}(h) = \frac{\sum_{h_1,\dots,h_k: f(h_1)+\cdots+f(h_k)=h} [\prod_{j=1}^k a(h_j)\, v_n(h_j)]}{\sum_{h_1,\dots,h_k} [\prod_{j=1}^k a(h_j)\, v_n(h_j)]} \tag{4.2}$$

Denote

$$s = \sinh h, \qquad s_j = \sinh h_j, \qquad t_j = \tanh h_j$$

We have

$$s^2 = \sinh^2 h = \sinh^2[f(h_1) + \cdots + f(h_k)] = \frac{\tanh^2[f(h_1) + \cdots + f(h_k)]}{1 - \tanh^2[f(h_1) + \cdots + f(h_k)]}$$

and

$$\tanh(x_1 + \cdots + x_k) = \frac{\sum_{\text{odd } p} \sum_{j_1 < j_2 < \cdots < j_p} y_{j_1} \cdots y_{j_p}}{\sum_{\text{even } p} \sum_{j_1 < j_2 < \cdots < j_p} y_{j_1} \cdots y_{j_p}}, \qquad y_1 = \tanh x_j$$

so

$$\tanh[f(h_1) + \cdots + f(h_k)] = \frac{\sum_{\text{odd } p} \theta^p \sum_{j_1 < j_2 < \cdots < j_p} t_{j_1} \cdots t_{j_p}}{\sum_{\text{even } p} \theta^p \sum_{j_1 < j_2 < \cdots < j_p} t_{j_1} \cdots t_{j_p}}$$

and

$$\begin{aligned} s^2 = \sinh^2 h &= \left(\frac{\sum_{\text{odd } p} \theta^p \sum_{j_1 < j_2 < \cdots < j_p} t_{j_1} \cdots t_{j_p}}{\sum_{\text{even } p} \theta^p \sum_{j_1 < j_2 < \cdots < j_p} t_{j_1} \cdots t_{j_p}} \right)^2 \\ &\quad \times \left[1 - \left(\frac{\sum_{\text{odd } p} \theta^p \sum_{j_1 < j_2 < \cdots < j_p} t_{j_1} \cdots t_{j_p}}{\sum_{\text{even } p} \theta^p \sum_{j_1 < j_2 < \cdots < j_p} t_{j_1} \cdots t_{j_p}} \right)^2 \right]^{-1} \\ &= \frac{(\sum_{\text{odd } p} \theta^p \sum_{j_1 < j_2 < \cdots < j_p} t_{j_1} \cdots t_{j_p})^2}{\prod_{j=1}^k (1 - \theta^2 t_j^2)} \end{aligned}$$

Using the symmetry of $v_n(h)$, we obtain that

$$\begin{aligned} D_{n-1} &= \sum_h \sinh^2 h v_{n-1}(h) \\ &= \frac{\begin{pmatrix} \sum_{h_1,\dots,h_k} (\sum_{\text{odd } p} \theta^{2p}) \sum_{j_1 < j_2 < \cdots < j_p} t_{j_1}^2 \cdots t_{j_p}^2) \\ \times \prod_{j=1}^k [(1 - \theta^2 t_j^2)^{-1} a(h_j)\, v_n(h_j)] \end{pmatrix}}{\sum_{h_1,\dots,h_k} [\prod_{j=1}^k a(h_j)\, v_n(h_j)]} \end{aligned}$$

Now we substitute

$$t_j^2 = \frac{s_j^2}{1+s_j^2}$$

and obtain

$$D_{n-1} = \frac{\left(\begin{array}{c}\sum_{h_1,\dots,h_k} [\sum_{\text{odd } p} \theta^{2p} \sum_{j_1<j_2<\dots<j_p} s_{j_1}^2 \cdots s_{j_p}^2 \prod_{j \neq j_1,\dots,j_p} (1+s_j^2)] \\ \times \prod_{j=1}^k a^{-1}(h_j)\, v_n(h_j)\end{array}\right)}{\sum_{h_1,\dots,h_k} [\prod_{j=1}^k a(h_j)\, v_n(h_j)]} \tag{4.3}$$

To estimate the RHS of (4.3) we use the FKG inequality.

Lemma 4.2 (FKG inequality). Assume that $b(x)$, $c(x)$, and $d(x)$ are functions on a finite set X such that

$$d(x) \geqslant 0 \qquad \forall x \in X$$

$$[b(x_1)-b(x_2)][c(x_1)-c(x_2)] \geqslant 0 \qquad \forall x_1, x_2 \in X$$

$$\sum_x d(x) > 0, \qquad \sum_x c(x)\, d(x) > 0$$

Then

$$\frac{\sum_x b(x)\, c(x)\, s(x)}{\sum_x c(x)\, d(x)} \geqslant \frac{\sum_x b(x)\, d(x)}{\sum_x d(x)} \tag{4.4}$$

Proof. We have

$$0 \leqslant \sum_{x_1,x_2} [b(x_1)-b(x_2)][c(x_1)-c(x_2)]\, d(x_1)\, d(x_2)$$

$$= 3 \sum_{x_1} b(x_1)\, c(x_1)\, d(x_1) \sum_{x_2} d(x_2) - 2 \sum_{x_1} b(x_1)\, d(x_1) \sum_{x_2} c(x_2)\, d(x_2)$$

which implies (4.4). ∎

Applying the FKG inequality to the RHS of (4.3), we obtain

$$D_{n-1} \leqslant \frac{\left(\begin{array}{c}\sum_{h_1,\dots,h_k} [\sum_{\text{odd } p} \theta^{2p} \sum_{j_1<j_2<\dots<j_p} s_{j_1}^2 \cdots s_{j_p}^2 \prod_{j \neq j_1,\dots,j_p} (1+s_j^2)] \\ \times \prod_{j=1}^k v_n(h_j)\end{array}\right)}{\cdot \quad \sum_{h_1,\dots,h_k} [\prod_{j=1}^k a^2(h_j)\, v_n(h_j)]}$$

which gives

$$D_{n-1} \leqslant \frac{\sum_{\text{odd } p} C_k^p \theta^{2p} D_n^p (1+D_n)^{k-p}}{[1+(1-\theta^2)\, D_n]^k}, \qquad C_k^p = \frac{k!}{p!\,(k-p)!} \tag{4.5}$$

Lemma 4.3. If $k\theta^2 \leqslant 1$, then

$$\frac{\sum_{\text{odd } p} C_k^p \theta^{2p} D_n^p (1+D_n)^{k-p}}{[1+(1-\theta^2)\, D_n]^k} \leqslant k\theta^2 D_n \tag{4.6}$$

Proof. Denote for a fixed $D_n > 0$,

$$A(\theta) = \frac{\sum_{\text{odd } p} C_k^p \theta^{2p} D_n^p (1+D_n)^{k-p}}{k\theta^2 D_n [1+(1-\theta^2)\, D_n]^k} = \frac{\sum_{\text{odd } p} C_k^p \theta^{2(p-1)} D_n^{p-1} (1+D_n)^{k-p}}{k[1+(1-\theta^2)\, D_n]^k}$$

Then (4.6) is equivalent to

$$A(\theta) \leqslant 1$$

Observe that $A(\theta)$ is an increasing function of θ, so it is sufficient to prove that

$$A(k^{-1/2}) \leqslant 1 \tag{4.7}$$

By the Newton binomial formula,

$$\begin{aligned} A(k^{-1/2}) &= \frac{\sum_{\text{odd } p} C_k^p k^{-p} D_n^p (1+D_n)^{k-p}}{D_n[1+(1-k^{-1})\, D_n]^k} \\ &= \frac{(1+D_n+k^{-1}D_n)^k - (1+D_n-k^{-1}D_n)^k}{2D_n[1+(1-k^{-1})\, D_n]^k} \\ &= \frac{\{[1+(1+k^{-1})\, D_n/[1+(1-k^{-1})\, D_n]\}^k - 1}{2D_n} \end{aligned} \tag{4.8}$$

Put

$$z = \frac{1+(1+k^{-1})\, D_n}{1+(1-k^{-1})\, D_n} - 1 = \frac{2D_n}{k+(k-1)\, D_n}$$

Then

$$0 < z < \frac{2}{k-1} \qquad \text{when} \quad 0 < D_n < \infty$$

In addition,

$$D_n = \frac{kz}{2-(k-1)z}$$

and (4.7) is equivalent [due to (4.8)] to

$$(1+z)^k - 1 \leqslant 2D_n = \frac{kz}{1-(k-1)z/2}$$

Expanding both sides of the last inequality in Taylor series in z, we reduce it to

$$\begin{aligned} z + \frac{k(k-1)}{2} z^2 + \frac{k(k-)(k-2)}{6} z^3 + \cdots \\ \leqslant kz + \frac{k(k-1)}{2} z^2 + \frac{k(k-1)^2}{4} z^3 + \cdots \end{aligned} \tag{4.9}$$

which is obvious. Thus (4.7) is proved and this finishes the proof of Lemma 4.3. ∎

From (4.5) and Lemma 4.3 we obtain (4.1). Lemma 4.1 is proved. ∎

Now we observe that (4.9) can be strengthened to

$$\begin{aligned} kz + \frac{k(k-1)}{2} z^2 + \frac{k(k-1)(k-2)}{6} z^3 + \cdots \\ \leqslant \frac{kz + [k(k-)/2] z^2 + [k(k-1)^2/4] z^3 + \cdots}{1 + [(k-1)^2/12] z^2} \end{aligned}$$

which leads to

$$(1+z)^k - 1 \leqslant \frac{2D_n}{1 + [(k-1)^2/12]\{2D_n/k + (k-1)\, D_n]\}^2}$$

and then we get the following result.

Lemma 4.4. We have

$$D_{n-1} \leqslant \frac{k\theta^2 D_n}{1 + [(k-1)^2/3]\{D_n/[k + (k-1)\, D_n]\}^2} \tag{4.10}$$

This is useful when $k\theta^2 = 1$.

5. *Proof of the Theorem.* The inequality (4.10) says that

$$\lim_{N-n \to \infty} D_n = 0 \tag{5.1}$$

for $0<\theta\leqslant 1/\sqrt{k}$. Following the line of reasoning of Section 3 from ref. 1, one can now check that (5.1) implies the extremality of the disordered state $\mu^{\#}$. This means that for any $\varepsilon>0$, $n>0$ and any configuration $\sigma_n\subset\Sigma(V_n)$, there exist $N>n$ and a set $\Omega_N\subset\Sigma(W_N)$ such that:

(i) $\mu^{\#}(\Omega_N)>1-\varepsilon$.

(ii) $|\mu^{\#}(\sigma_n\mid\sigma^{(N)})-\mu^{\#}(\sigma_n)|<\varepsilon,\ \forall\sigma^{(N)}\in\Omega_N$.

This gives our main result. ∎

One of the open problems concerns the purity of the limiting Gibbs states in the case of nonzero external field. Another problem is the characterization of the limiting states for the random field model. The ground states for this model for binary distribution were examined in ref. 8.

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REFERENCES

1. P. M. Bleher, Extremity of the disordered phase in the Ising model on the Bethe lattice, *Commun. Math. Phys.* **128**:411–419 (1990).
2. T. Moore and J. L. Snell, A branching process showing a phase transition, *J. Appl. Prob.* **16**:252–260 (1979).
3. H.-O. Georgii, *Gibbs Measures and Phase Transitions* (De Gruyter, Berlin, 1988).
4. Y. Higuchi, Remarks on the limiting Gibbs states on a $(d+1)$-tree, *Publ. RIMS Kyoto Univ.* **13**:335–348 (1977).
5. P. M. Bleher and N. N. Ganihodgaev, On pure phases of the Ising model on the Bethe lattice, *Theory Prob. Appl.* **35**:216–227 (1990).
6. P. M. Bleher, Letter to H.-O. Georgii, January 21, 1992.
7. J. M. Carlson, J. T. Chayes, L. Chayes, J. P. Sethna, and D. J..Thouless, Critical behavior of the Bethe lattice spin-glass, *Europhys. Lett.* **5**:355–360 (1989).
8. R. Bruinsma, Random field Ising model on a Bethe lattice, *Phys. Rev. B* **30**:289–299 (1984).

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